Oct 31, 2022

Week 9

2020A Adv. Cal. II

Last time we defined

$$\int_{\vec{C}} f\,ds$$

for a regular parametric curve $\vec{C}(t)$, $t \in [a,b]$, to be

$$\int_a^b f(\vec{C}(t))\,|\vec{C}'(t)|\,dt.$$

Now, we define a <u>curve</u> to be a subset C in $\mathbb{R}^2, \mathbb{R}^3$ so that it is the image of a regular parametric curve. The line integral of f along a curve C is defined to be

$$\int_C f\,ds = \int_a^b f(\vec{C}(t))\,|\vec{C}'(t)|\,dt.$$

One can see from the Riemann sum approach to the line integral this integral is independent of the choice of the parametrization.

Also, we can add two curves C_1, C_2 together to form $C = C_1 + C_2$ if the endpoints of C_1, C_2 match, that is,

C_1 described by $\vec{C}_1(t)$, $t \in [a,b]$,
C_2 described by $\vec{C}_2(t)$, $t \in [c,d]$,
$\vec{C}_1(b) = \vec{C}_2(c)$.

Define

$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds.$$

e.g. Evaluate $\int_C (x-3y^2+z)\,ds$ where $C = C_1 + C_2$,

C_1 is the line segment bet. $(0,0,0)$ and $(1,1,0)$

C_2 is - - - - - - - - bet $(1,1,0)$ and $(1,1,1)$

Choose parametrization for C_1 & C_2:

$$C_1 : \vec{c}_1(t) = (0,0,0) + t((1,1,0)-(0,0,0))$$
$$= (t,t,0), \quad t\in[0,1]$$
$$C_2 \quad \vec{c}_2(t) = (1,1,0) + t((1,1,1)-(1,1,0))$$
$$= (1,1,0) + (0,0,t)$$
$$= (1,1,t), \quad t\in[0,1]$$

$$\int_{C_1} (x-3y^2+z)\,ds = \int_0^1 (t-3t^2+0)\,|c_1'(t)|\,dt$$
$$= \int_0^1 (t-3t^2)\sqrt{2}\,dt = -\frac{\sqrt{2}}{2}$$

$$\int_{C_2} (x-3y^2+z)\,ds = \int_0^1 (1-3+t)\sqrt{1}\,dt = -\frac{1}{2}$$

$$\int_C (x-3y^2+z)\,ds = -\frac{\sqrt{2}}{2} - \frac{3}{2}. \quad \#$$

Line integral of a v.f.

$$\vec{F} = (M,N,P) = M\hat{i} + N\hat{j} + P\hat{k} \quad - \mathbb{R}^3$$
$$= (M,N) = M\hat{i} + N\hat{j} \quad - \mathbb{R}^2$$

Consider a curve from $\vec{P}$ to $\vec{Q}$, $\vec{c}(t)$, $t\in[a,b]$, $\vec{c}(a) = \vec{P}$, $\vec{c}(b) = \vec{Q}$. Define

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{c}(t))\cdot\vec{c}\,'(t)\,dt$$

~ When $\vec{F}$ is the force field, $\int_C \vec{F}\cdot d\vec{r}$ gives the work done of $\vec{F}$ along C.

~ $\int_C \vec{F}\cdot d\vec{r}$ is the same as long as the parametrization runs from $\vec{P}$ to $\vec{Q}$.

~ $\int_C \vec{F}\cdot d\vec{r}$ gets a "$-$" sign when the parametrization runs from $\vec{Q}$ to $\vec{P}$.

Thus when we evaluate $\int_C \vec{F}\cdot d\vec{r}$ we need to specify the orientation of the curve. A curve with a specific orientation is called an oriented curve. For a closed curve, the orientation is either anticlockwise or clockwise.

Another notation

$$\int_C \vec{F}\cdot d\vec{r} = \int_C M\,dx + N\,dy + P\,dz \qquad (\mathbb{R}^3)$$

$$= \int_C M\,dx + N\,dy \qquad (\mathbb{R}^2)$$

e.g. Evaluate $\int_C \vec{F}\cdot d\vec{r}$, $\vec{F} = z\hat{i} + xy\hat{j} - y^2\hat{k}$
C, $\vec{r}(t) = t^2\hat{i} + t\hat{j} + \sqrt{t}\,\hat{k}$, $0 \le t \le 1$

$$\vec{r}\,'(t) = 2t\,\hat{i} + \hat{j} + \frac{1}{2\sqrt{t}}\hat{k}$$

$$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = \sqrt{t}\,2t + t^2 \times t \times 1 - t^2\frac{1}{2\sqrt{t}}$$

$$= 2t^{3/2} + t^3 - \frac{1}{2}t^{3/2}$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_0^1 (2t^{3/2} + t^3 - \tfrac{1}{2}t^{3/2})\,dt = \cdots = \frac{17}{20}. \#$$

e.g. Evaluate

$$\int_C -y\,dx + z\,dy + 2x\,dz, \quad C = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k}, \quad t \in [0, 2\pi]$$

$$\int_C -y\,dx + z\,dy + 2x\,dz = \int_0^{2\pi} (-\sin t\,\hat{i} + t\,\hat{j} + 2\cos t)\cdot(-\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k})\,dt$$

$$= \int_0^{2\pi} (\sin^2 t + t\cos t + 2\cos t)\,dt$$

$$= \pi \quad \#$$

Another point of view other than work done.

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{c}(t))\cdot\vec{c}\,'(t)\,dt$$

$$= \int_a^b \vec{F}(\vec{c}(t))\cdot\frac{\vec{c}\,'(t)}{|\vec{c}\,'(t)|}\,|\vec{c}\,'(t)|\,dt$$

$$= \int_C \vec{F}\cdot\hat{t}\,ds, \qquad \hat{t} = \frac{\vec{c}\,'(t)}{|\vec{c}\,'(t)|} \quad \text{unit tangent}$$

$\vec{F}\cdot\hat{t}$ is the projection of $\vec{F}$ onto the tangential direction of the curve. When $\vec{F}$ stands for the velocity of some fluid, this integral give the amount of flow along the curve C. When C is a closed one, this flow is called the <u>circulation</u>.

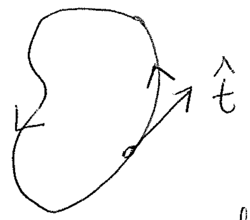
e.g. Find the flow of $\vec{F} = (x, z, y)$ along $\vec{r}(t) = (\cos t, \sin t, t)$, $t \in [0, \pi/2]$.

$$\text{flow} = \int_0^{\pi/2} (\cos t, t, \sin t)\cdot(-\sin t, \cos t, 1)\,dt$$

$$= \int_0^{\pi/2} (-\cos t\sin t + t\cos t + \sin t)\,dt$$

$$= \frac{\pi - 1}{2} \quad \#$$

When C is a closed curve in $\mathbb{R}^2$ running in anticlockwise way, its tangent is



$$\hat{t} = \frac{(x'(t), y'(t))}{\sqrt{x'^2(t)+y'^2(t)}}$$

the unit outer normal

$$\hat{n} = \frac{(y'(t), -x'(t))}{\sqrt{x'^2(t)+y'^2(t)}}.$$

The <u>flux</u> of $\vec{F}$ across C is

$$\int_C \vec{F}\cdot\hat{n}\, ds.$$

As $\vec{F} = M\hat{i}+N\hat{j}$,

$$\int_C \vec{F}\cdot\hat{n}\, ds = \int_a^b \left(M(\vec{c}(t))\frac{y'(t)}{\sqrt{\quad}} - N(\vec{c}(t))\frac{x'(t)}{\sqrt{\quad}}\right)\sqrt{\quad}\, dt$$

$$= \int_a^b M(\vec{c}(t))y'(t) - N(\vec{c}(t))x'(t)\, dt$$

$$= \int_C M\,dy - N\,dx,$$

this is another form for the flux.

e.g. Find the flux of the velocity v.f $(x-y, x)$ across the unit circle $x^2+y^2=1$.

First, we choose $t \mapsto (\cos t, \sin t)$, $t \in [0, 2\pi]$ to parametrize the circle, $\hat{t} = (-\sin t, \cos t)$, $\hat{n} = (\cos t, \sin t)$

$$\text{flux} = \int_0^{2\pi} (\cos t - \sin t, \cos t)\cdot(\cos t, \sin t)\, dt$$

$$= \int_0^{2\pi} \cos^2 t\, dt = \pi. \quad \#$$

(Cont'd)

We recall some definitions.

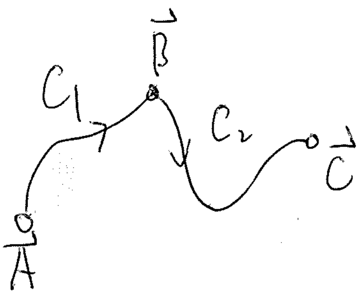
A parametric curve is a map $\vec{c} : [a,b] \to \mathbb{R}^2, \mathbb{R}^3$ where each components $x(t), y(t), z(t)$ are continuous. It is regular if $x(t), y(t), z(t)$ are continuously differentiable and

$$|\vec{c}'(t)| = (x'(t)^2 + y'(t)^2 + z'(t)^2) > 0, \quad \forall t \in (a,b).$$

A curve is the image of some parametric curve. It is a smooth curve if the parametization is a regular one.

Let C_1 be a smooth curve from $\vec{A}$ to $\vec{B}$, and C_2 a smooth curve from $\vec{B}$ to $\vec{C}$. Then $C = C_1 + C_2$ is a piecewise smooth curve from $\vec{A}$, to $\vec{C}$

$$\int_C \vec{F} \cdot d\vec{r} \overset{\text{def}}{=} \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} .$$



A vector field $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$ is usually defined in an open region. It is usually assumed to be continuous, ie, M, N, P are continuous functions in the region. It is called a smooth v.f. if M, N, P are continuously differentiable.

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A v.f. $\vec{F}$ in open region G is conservative if there is a function Φ in G s.t. $\nabla \Phi = \vec{F}$, ie

$$\frac{\partial \Phi}{\partial x} = M, \quad \frac{\partial \Phi}{\partial y} = N, \quad \frac{\partial \Phi}{\partial z} = P .$$

(When $\vec{F}$ defined in 2-dim, $\frac{\partial \Phi}{\partial x} = M$, $\frac{\partial \Phi}{\partial y} = N$.) The function Φ is the potential for $\vec{F}$.

<u>Theorem</u> For a conservative v.f. $\vec{F}$ in G,

$$\int_C \vec{F}\cdot d\vec{r} = \Phi(\vec{Q}) - \Phi(P)$$

where Φ is the potential of $\vec{F}$, C is a piecewise smooth curve from $\vec{P}$ to $\vec{Q}$ in G.

Pf: $\int_C \vec{F}\cdot d\vec{r} \overset{\text{def}}{=} \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$ when $\vec{r}$ is a regular parametrization of C (assume C to be smooth).

$$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = M(\vec{r}(t))\,x'(t) + N(\vec{r}(t))\,y'(t) + P(\vec{r}(t))\,z'(t).$$

On the other hand,

$$\Phi(\vec{r}(t)) = \Phi(x(t), y(t), z(t)).$$

By the Chain Rule,

$$\begin{aligned}\frac{d}{dt}\Phi(\vec{r}(t)) &= \frac{d}{dt}\Phi(x(t), y(t), z(t)) \\ &= \frac{\partial\Phi}{\partial x}(\vec{r}(t))\,x'(t) + \frac{\partial\Phi}{\partial y}(\vec{r}(t))\,y'(t) + \frac{\partial\Phi}{\partial z}(\vec{r}(t))\,z'(t) \\ &= M(\vec{r}(t))\,x'(t) + N(\vec{r}(t))\,y'(t) + P(\vec{r}(t))\,z'(t)\end{aligned}$$

$$\begin{aligned}\therefore \int_C \vec{F}\cdot d\vec{r} &= \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt \\ &= \int_a^b \frac{d}{dt}\Phi(\vec{r}(t))\,dt \\ &= \Phi(\vec{r}(t))\Big|_a^b = \Phi(\vec{r}(b)) - \Phi(\vec{r}(a)) = \Phi(\vec{Q}) - \Phi(\vec{P}).\end{aligned}$$

When C is piecewise smooth, for example, $C = C_1 + C_2$, C_1 from $\vec{P}$ to $\vec{R}$, and C_2 from $\vec{R}$ to $\vec{Q}$. As both C_1 and C_2 are smooth, by the previous discussion,

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \Phi(\vec{R}) - \Phi(\vec{P}), \quad \int_{C_2} \vec{F}\cdot d\vec{r} = \Phi(\vec{Q}) - \Phi(R)$$

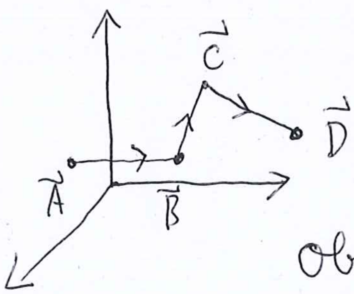
I forgot to mention in class

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$$

$$= \Phi(\vec{R}) - \Phi(\vec{P}) + \Phi(\vec{Q}) - \Phi(R)$$

$$= \Phi(\vec{Q}) - \Phi(\vec{P}). \quad \#$$

e.g. Find the work done of the force $-\frac{g}{z^2}\hat{k}$ along the path $C = C_1 + C_2 + C_3$



C_1: line from $(1,0,1)$ to $(1,1,1)$

C_2: line from $(1,1,1)$ to $(1,2,3)$,

C_3: line from $(1,2,3)$ to $(0,1,\frac{1}{2})$.

Observe that $\vec{F} = 0\hat{i} + 0\hat{j} - \frac{g}{z^2}\hat{k}$ has a potential $\Phi(x,y,z) = \frac{g}{z}$ in the upper half space.

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \Phi(0,1,\tfrac{1}{2}) - \Phi(1,0,1)$$

$$= \frac{g}{\frac{1}{2}} - \frac{g}{1} = g. \quad \#$$

Therefore, it is desirable to determine when a v.f is conservative.

Component Test If v.f $\vec{F}$ is conservative. then

$$M_y = N_x, \quad M_z = P_x, \quad N_z = P_y$$

(when $n=2$, $M_y = N_x$)

Pf: If $\vec{F} = \nabla\Phi$, ie, $M = \Phi_x$, $N = \Phi_y$, $P = \Phi_z$. Then

$M_y = \Phi_{xy}$, $N_x = \Phi_{yx}$. As $\Phi_{xy} = \Phi_{yx}$, $M_y = N_x$.

Similarly, get other 2 relations.

eg. Is $\vec{F} = xy\hat{i} + y^2\hat{j} + zy\hat{k}$ conservative?

As $M_y = x$, $N_x = 0$, $M_y \neq N_x$, so $\vec{F}$ is not conservative.

e.g. Find (if any) a potential for

$$\vec{F} = (e^x\cos y + yz)\hat{i} + (xz - e^x \sin y)\hat{j} + (xy + z)\hat{k}.$$

Component test first.

$$M_y = \frac{\partial}{\partial y}(e^x \cos y + yz) = -e^x \sin y + z$$

$$N_x = \frac{\partial}{\partial x}(xz - e^x \sin y) = z - e^x \sin y. \quad \therefore M_y = N_x.$$

$$M_z = y, \quad P_x = y \quad \therefore M_z = P_x.$$

$$N_z = x, \quad P_y = x \quad \therefore N_z = P_y.$$

$\vec{F}$ passes the component test. Most likely it has a potential.

$$\frac{\partial\Phi}{\partial x} = M = e^x\cos y + yz$$

$\therefore \quad \Phi(x,y,z) = e^x \cos y + xyz + g(y,z)$, g to be determined.

$$\frac{\partial\Phi}{\partial y} = -e^x \sin y + xz + \frac{\partial g}{\partial y}(y,z)$$

$$= N = xz - e^x \sin y \Rightarrow \frac{\partial g}{\partial y}(y,z) = 0, \therefore g(y,z) = h(z)$$

$$\frac{\partial\Phi}{\partial z} = xy + h'(z) = P = xy + z \Rightarrow h'(z) = z, \therefore h(z) = \frac{z^2}{2} + C.$$

Conclusion: $\Phi(x,y,z) = e^x \cos y + xyz + \frac{z^2}{2} + C$, C constant.

Remark. There are examples showing that even $\vec{F}$ passes the component test, a potential does not exist. We will discuss more next week.